Exact general relativistic lensing versus thin lens approximation: the crucial role of the void

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We have used an exact general relativistic model structure within a FRW cosmological background based on a LTB metric to study the gravitational lensing of a cosmological structure. The integration of the geodesic equations turned out to be a delicate task. We realized that the use of the rank 8(7) and 10(11) Runge-Kutta numerical method leads to a numerical effect and is therefore unreliable. The so-called semi-implicit Rosenbrock method, however, turned out to be a viable integration method for our problem. The deviation angle calculated by the integration of the geodesic equations for different density profiles of the model structure was then compared to those of the corresponding thin lens approximation. Using the familiar NFW density profile, it is shown that independent of the truncation details the thin lens approximation differ substantially from the exact relativistic calculation. The difference in the deflection angle for different impact parameters may be up to about 30 percent. However, using the modified NFW density profile with a void before going over to the FRW background, as required by an exact general relativistic model, the thin lens approximation coincides almost exactly with the general relativistic calculation.

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The thin lens (Th-L) approximation in the gravitational lensing is the prevailing method to estimate cosmological parameters and the mass of large scale structures leading to dark matter and dark energy contents of the universe [1], [2]. The current view is that this Th-L approximation is accurate enough at the cosmological scales where we are faced with very weak gravitational fields and potentials. There has already been attempts to compare the Th-L approximation with the integration of null geodesics in a perturbed cosmological background ([3–5], see also [5] and the references there). However, a full general relativistic calculation based on an exact model is still missing. There are two sources of misinterpretation of astrophysical phenomenon in a weak gravity environment, depending on the local or quasi-local phenomena under consideration. In the case of local phenomena the familiar perturbation theories maybe valid to some extend. There are already detailed studies on this subject (see [6], [7], [8]). However, if quasi-local phenomena or structures come into play we may encounter counter-intuitive effects not detected in the perturbational approach to the weak field limits. The definition of quasi-local mass in general relativity is one of these issues which has been extensively studied in general relativity [9]. We have already shown numerically how different various quasi-local mass definitions of a general relativistic structure may be [10]. Another quasi-local effect relevant to the gravitational lensing is how a spherically symmetric structure is matched to a FRW background. Such a general relativistic matching is only possible through an underdensity region or a void [11]; a fact not realized in the post-Newtonian approaches or cosmological perturbations relevant to lensing, and missed in all studies comparing the Th-L approaches to a more exact general relativistic lensing calculation.

We are interested in the exact general relativistic lensing by an exact solution of Einstein Equations representing a cosmological structure defined by a spherically symmetric overdensity structure within a FRW universe. There is already an exact general relativistic model structure within an FRW universe based on a Lemaître, Tolman and Bondi (LTB) metric [13–15] representing an inhomogeneous cosmological model with a structure at its center[10]. Choosing such a model for an extended spherical lens, we study the gravitational lensing in a dynamical cosmological background in the framework of general relativity by integrating numerically the null geodesic equations to obtain the deflection angle. The result is then compared with the corresponding Th-L approximation to understand the accuracy of this technology and its possible flaws in interpreting the structure and the

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mass of cluster of galaxies. The effect of the cosmological constant in the lensing is negligible in small scales we are considering [12] and only effect the cosmological distances which we will take into account. That is why we have neglected the cosmological constant in our exact model to avoid unnecessary complexities

Take a spherically symmetric cosmological structure in a FRW matter dominated universe with the density $\rho(r,t)$. This is modeled by a LTB solution of the Einstein equations which is written in the comoving coordinates as $(G = 1, c = 1)$

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + R^2(t,r)d\Omega^2. \quad (1)$$

satisfying

$$\rho(r,t) = \frac{M'(r)}{4\pi R^2 R'}, \quad (2)$$

$$X = \frac{R'}{\sqrt{1 + E(r)}}, \quad (3)$$

$$\dot{R}^2 = E(r) + \frac{2M(r)}{R}. \quad (4)$$

Here M and E are integrating functions, where dot and prime denote partial derivatives with respect to the coordinates t and r respectively. Equation (4) has three different analytic solution, depending on the value of E. The solution for negative E we are interested in is given by

$$R = -\frac{M}{E}(1 - \cos\eta),$$

$$\eta - \sin\eta = \frac{(-E)^{3/2}}{M}(t - t_b(r)). \quad (5)$$

The solution has three free functions: $t_b(r)$, $E(r)$, and $M(r)$. Given that the metric is covariant under the rescaling $r \to \tilde{r}(r)$ one of these functions may be fixed. The geodesic equations may be written in the arbitrary plane of $\theta = \frac{\pi}{2}$ due to the spherical symmetry:

$$t: \frac{d^2t}{d\lambda^2} + X\dot{X}\left(\frac{dr}{d\lambda}\right)^2 + R\dot{R}\left(\frac{d\phi}{d\lambda}\right)^2 = 0, \quad (6)$$

$$r: \frac{d^2r}{d\lambda^2} + 2\frac{\dot{X}}{X}\frac{dr}{d\lambda}\frac{dt}{d\lambda} + \frac{X'}{X}\left(\frac{dr}{d\lambda}\right)^2 - \frac{RR'}{X^2}\left(\frac{d\phi}{d\lambda}\right)^2 = 0, \quad (7)$$

$$\phi: \frac{d^2\phi}{d\lambda^2} + 2\frac{\dot{R}}{R}\frac{dt}{d\lambda}\frac{d\phi}{d\lambda} + 2\frac{R'}{R}\frac{dr}{d\lambda}\frac{d\phi}{d\lambda} = 0, \quad (8)$$

where λ is an affine parameter. Equation (8) expresses the conservation of the angular momentum:

$$L = R^2\frac{d\phi}{d\lambda} = Const. \quad (9)$$

We are interested in the light-like geodesics. From the metric we obtain the light-like condition in the form

$$\left(\frac{dt}{d\lambda}\right)^2 = X^2\left(\frac{dr}{d\lambda}\right)^2 + R^2\left(\frac{d\phi}{d\lambda}\right)^2 \quad (10)$$

These partial non-linear differential equations can not be solved analytically. To integrate them numerically one has to specify the three functions $M(r)$, $t_b(r)$, and $E(r)$ and all derivatives of the metric functions, using a procedure proposed in [16, 17]. We start with a generic density profile and specify it at two different times t_1, t_2 as a function of the coordinate r. Now, the numerical procedure is based on the choice of r-coordinate such that $M(r) = r$. This is due to the fact that $M(r)$ is an increasing function of r. Therefore, E and t_b become functions of M. For the initial time we choose the time of the last scattering surface: $t_1 \simeq 3.77 \times 10^5 yr$. The initial density profile should show a small over-density near the center imitating otherwise a FRW universe. Therefore, we add a Gaussian peak to the FRW background density ρ_b. We know already that having an over-density in an otherwise homogeneous universe needs a void to compensate for the extra mass within the over-density region. Therefore, to model this void we subtract a wider gaussian peak:

$$\rho(R, t_1) = \rho_b(t_1)\left[\left(\delta_1 e^{-\left(\frac{R}{R_0}\right)^2} - b_1\right)e^{-\left(\frac{R}{R_1}\right)^2} + 1\right], \quad (11)$$

where δ_1 is the density contrast of the Gaussian peak, R_0 is the width of the Gaussian peak, and R_1 is the width of the negative Gaussian profile. The mass compensation condition leads to an equation for b_1. For the final time we choose the time when our null geodesy has the nearest distance to the center of our model structure. For instance if we set our lens at the redshift $z \simeq 0.2$ then $t_2 \simeq 6.98Gyr$.

The density profile we choose for the final time is the universal halo density profile (NFW) [18] convolved with a negative Gaussian profile to compensate the mass plus the background density at that time:

$$\rho(R, t_2) = (\rho_{NFW} - b_2\rho_b(t_2))e^{-\left(\frac{R}{R_2}\right)^2} + \rho_b(t_2), \quad (12)$$

where

$$\rho_{NFW} = \rho_b(t_2)\frac{\delta_c}{\left(\frac{R}{R_s}\right)\left(1 + \frac{R}{R_s}\right)^2} \quad (13)$$

and

$$\delta_c = \frac{200}{3}\frac{c^3}{\ln(1+c) - \frac{c}{1+c}}. \quad (14)$$

In our numerical calculation we will use typical NFW values $R_s = 0.5Mpc$ and $c = 5$ for a galaxy cluster. Note that at the time t_2 a black hole singularity covered by an apparent horizon has already been evolved. Therefore, the NFW profile has to be modified and a black hole mass greater than a minimum value has to be added to it at the center. This physical fact is reflected in a shell crossing singularity if we take the familiar NFW profile similar to that assumed for the time t_1. The mass we have assumed for this black hole singularity is about one thousandth of
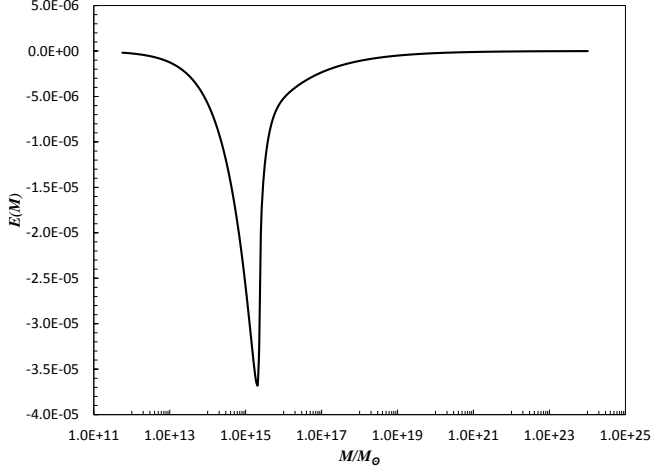
FIG. 1: E as a function of M for a cluster with NFW density profile. M is given in the unit of the Sun mass.
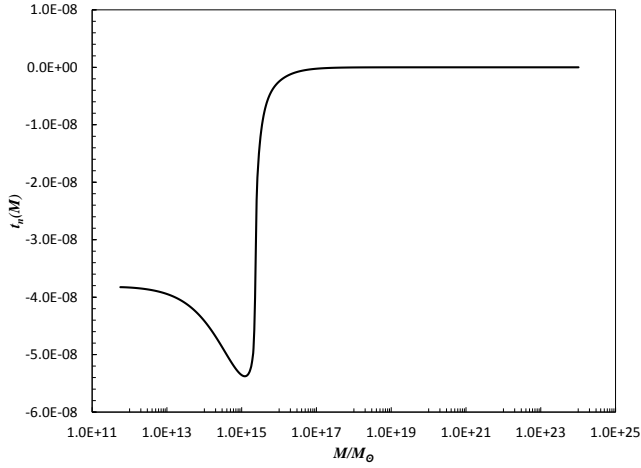

FIG. 2: t_b as a function of M for a cluster with NFW density profile. t_b is given in the unit of $3.263 Gyr$.

the mass up to the R_s and equal to $5.66 \times 10^{11} M_\odot$. Figs. 1 and 2 shows the LTB functions E and t_b as a result of these boundary assumptions. Using these LTB functions, the density profile of our model structure is obtained and depicted in Fig. 3.

To solve these equations we have to specify four initial conditions taking into account the light-like condition (10). The freedom of choosing the affine parameter reduces the initial conditions to three. Now, the integration of the geodesics happens by a backshooting procedure. Our initial conditions are taken to be the time of observation, distance of the observer to the lens expressed in terms of the redshift of the lens at the time of the observation, and angle between the line of sight to the image of source and the line of sight to the lens (θ in Fig. 4):
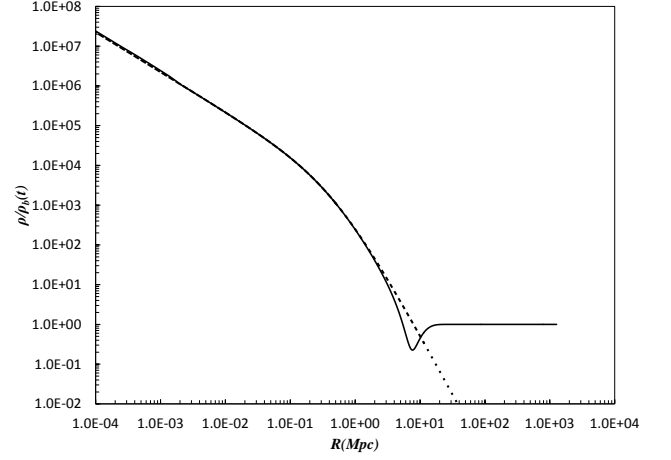

FIG. 3: Density profile for a cluster. The dot line corresponds to the familiar NFW profile and the solid line corresponds to the modified NFW with a void.

$$\tan\theta|_O = \left. \frac{R \frac{d\phi}{d\lambda}}{R' \frac{dr}{d\lambda}} \right|_{\text{Null}}. \qquad (15)$$

The integration is done from the observer to the source at a specific redshift. Assuming there is no lens, the model reduces to a homogenous flat FRW universe and the geodesics are straight lines (in comoving coordinates) allowing us to determine the angle between the source and the lens (β in Fig. 4):

$$\tan\beta = \frac{\sin\phi_e}{\frac{r_o}{r_e} - \cos\phi_e}, \qquad (16)$$

where ϕ_e is the $\widehat{OLS}$ angle, r_o is the comoving distance of the observer, and r_e is the comoving distance of the source from the center of coordinate system in the absence of lens at the time t_e. From the geodesic equations the t_e is given by

$$\left(t_o^{\frac{1}{3}} - t_e^{\frac{1}{3}}\right)^2 = \frac{1}{9}\left[\frac{R_o^2}{t_o^{\frac{4}{3}}} + \frac{R_e^2}{t_e^{\frac{4}{3}}} - \frac{2R_o R_e}{t_o^{\frac{2}{3}} t_e^{\frac{2}{3}}}\cos\phi_e\right]. \qquad (17)$$

We then write the lens equation and determine the deflection angle γ:

$$\gamma = (\theta - \beta)\frac{D_{OS}}{D_{LS}}, \qquad (18)$$

where we have assumed that the presence of the lens has not a significant effect on the distances and we may use the corresponding FRW ones.

The validity of the numerical method chosen to integrate such complex system of partial differential
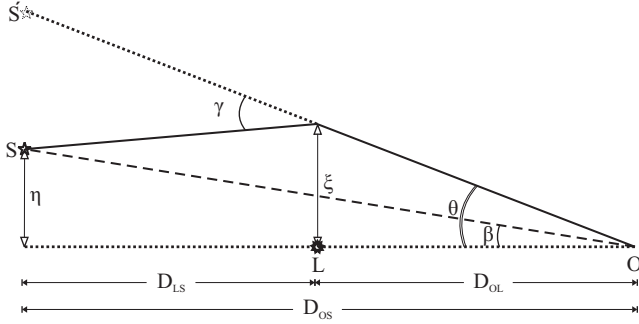
FIG. 4: GL diagram: O is observer, S is source, S' is image in source plane, L is lens and γ is deflection angle.

equations is a delicate issue. We first started with the familiar Runge-Kutta adaptive step size algorithm with proportional and integral feedback (PI control) [19] in which the step size is adjusted to keep local error under a suitable threshold. We started with the so-called embedded Runge-Kutta of the rank 5(4). It turned out, however, that its accuracy is too low. Therefore, we tried the rank 8(7) and then the rank 11(10) algorithm. The difference between these two last ranks, however, turned out to be marginal and below one percent. Given the time-consuming rank 11(10) algorithm, we preferred to use the rank 8(7) one. Now, as a fist test for the accuracy of this numerical method we tried the trivial example of the LTB model, namely the FRW case, expecting a null result. The result was a non-negligible deflection angle of the order of few milliarcseconds. Suspecting to face a numerical effect, and trying to understand the numerical algorithm and the source of this numerical effect, we continued to calculate a more concrete and non-trivial LTB case. The result for the rank 8(7) Runge-Kutta numerical method applied to a structure with a compact density profile did agree with the thin lens approximation. However, in the case of a more diffuse density profile the result showed a deflection angle up to an order of magnitude higher than the thin lens approximation. We did interpret this result as a sign not to trust the Runge-Kutta method and turned to an alternative numerical method!

The root of this numerical deficiency could be due to the term $\frac{d\phi}{d\lambda}$ in our equations, which is almost zero in the most part of the path of the light ray and changes suddenly to π in the vicinity of the lens. This is a well-known phenomenon in the numerical method of integrating differential equations called as "stiff" [20]. The characteristic property of such stiff equations is the presence of two quite different scales. In our case we have on one side the cosmological distance scale of the source relative to the lens and the observer, and on the other side the scale of the structure or the nearest

distance of the ray to the lens. Realizing this stiffness property, we turned to the so-called semi-implicit Rosenbrock method of the numerical integration of partial differential equations [19, 20]. As a first test we calculated again the trivial case of a FRW model which gave an acceptable null result. We, therefore, decided to integrate our geodesic equations using the semi-implicit Rosenbrock method instead of the Runge-Kutta one.

The null geodesics equations of our exact general relativistic structure model is now integrated using the modified NFW density profile with a void before matching to the background FRW universe to obtain the deflection angle. Note that the density in the NFW density profile is taken to be the oversdensity in an otherwise FRW model, namely $\rho - \rho_b$. However, for the Th-L approximation we have used two different density profiles namely the familiar one and the modified one with a void before matching to the background density. In the case of familiar NFW density profile without a void, the corresponding equations can be integrated analytically to give the deviation angle [24, 25]:

$$\gamma(x) = \frac{4M_{sing}}{xR_s} + 16\pi\rho_b\delta_c\frac{R_s^2}{x}\left(\log\frac{x}{2} + F(x)\right) \quad (19)$$

$$F(x) = \begin{cases} \frac{\text{arctanh}(\sqrt{1-x^2})}{\sqrt{1-x^2}} & x < 1 \\ 1 & x = 1 \\ \frac{\arctan(\sqrt{x^2-1})}{\sqrt{x^2-1}} & x > 1 \end{cases} \quad (20)$$

Assuming the same modified NFW profile as in general relativistic case for the Th-L approximation we have also calculated the deflection angle applying the lens equation [1]

$$\theta - \beta = \frac{D_{LS}}{D_{OL}D_{OS}}\frac{d\Psi(\theta)}{d\theta}, \quad (21)$$

where Ψ is the lens potential.

The result for the three cases, the exact general relativistic model with our modified NFW profile, thin lens approximation using the modified NFW with void, and the thin lens approximation using the familiar NFW without a void is depicted in Fig. 5. Obviously the two cases of the thin lens approximation with the modified NFW density profile including the void and the LTB exact method almost coincide. The thin lens approximation with the familiar density profile without a void, however, differ from the exact LTB model. The difference in the deviation angle can be more than 30 percent depending on the impact parameter. The difference between the exact general relativistic LTB model and the thin lens approximation is due to the absence of the void in the familiar NFW profile used in the literature. To see the implications of the NFW parameters in this difference we have also calculated the deviation angle for different NFW profiles, with and without void. The result is depicted in the Fig. 19. We see again that the Th-L approximation using different modified NFW profiles including
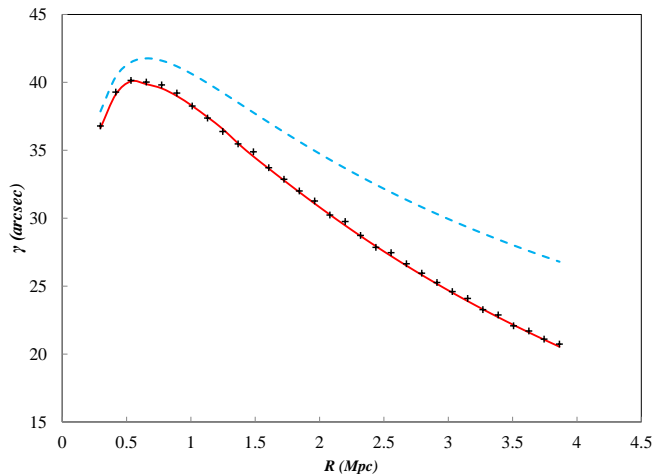
FIG. 5: Deviation angle for three cases: the general relativistic result is indicated by plus points; the thin lens approximation using our modified NFW is shown by the continuous line; and the dashed line is for the familiar NFW profile without the void (formula (19)).
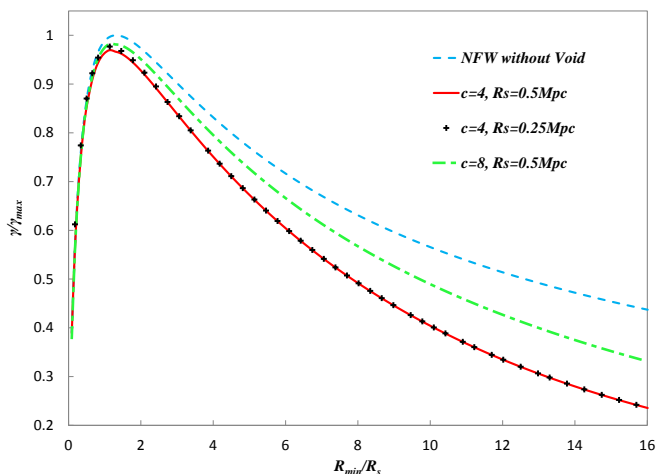

FIG. 6: Deviation angle for NFW density profiles with different parameters. Horizontal axis is normalized to R_s and vertical axis is normalized to the maximum of the deflection angle in each case. Dash line is for NFW model without void (formula (19)).

a void almost coincide with the exact LTB model. Models with the NFW profiles without void, however, differ substantially from the exact model. The difference is higher the bigger the c_s parameter is, i.e. the less the concentration of the density of structure is.

We, therefore, conclude that by interpreting astrophysical data of gravitational lensing by clusters using a familiar NFW density profile without a void we are deviating from the exact result and the Th-L approximation is no longer valid. The Th-L approximation may, however, be considered as precise enough if one modify the density profile and add the corresponding void to it, as require by general relativity for a quasi-local structure. The detail of the void, such as its density contrast, its depth and length, depends on the detail of the model and the deviation from the familiar NFW may even be much higher for other choices. Also note that the effect of the void is higher for larger impact parameter. In the case of strongly lensed objects in astrophysical applications we are usually faced with small impact parameter where this effect is negligible. For example in the case of Abell 2261 cluster ($z = 0.225$) with many strong lensing arcs, D. Coe et al. [27] have assigned $c_s = 6.2 \pm 0.3$ and $M_{vir} = 2.2 \pm 0.2 \times 10^{15} M_{\odot}$. The exact general relativistic results according to our model would lead to $c_s = 6.23$ and $M_{vir} = 2.23 \times 10^{15} M_{\odot}$. In the case of weak lensing, however, we expect this effect to have significant impact on the cosmological parameters. Work in this direction is in progress.

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